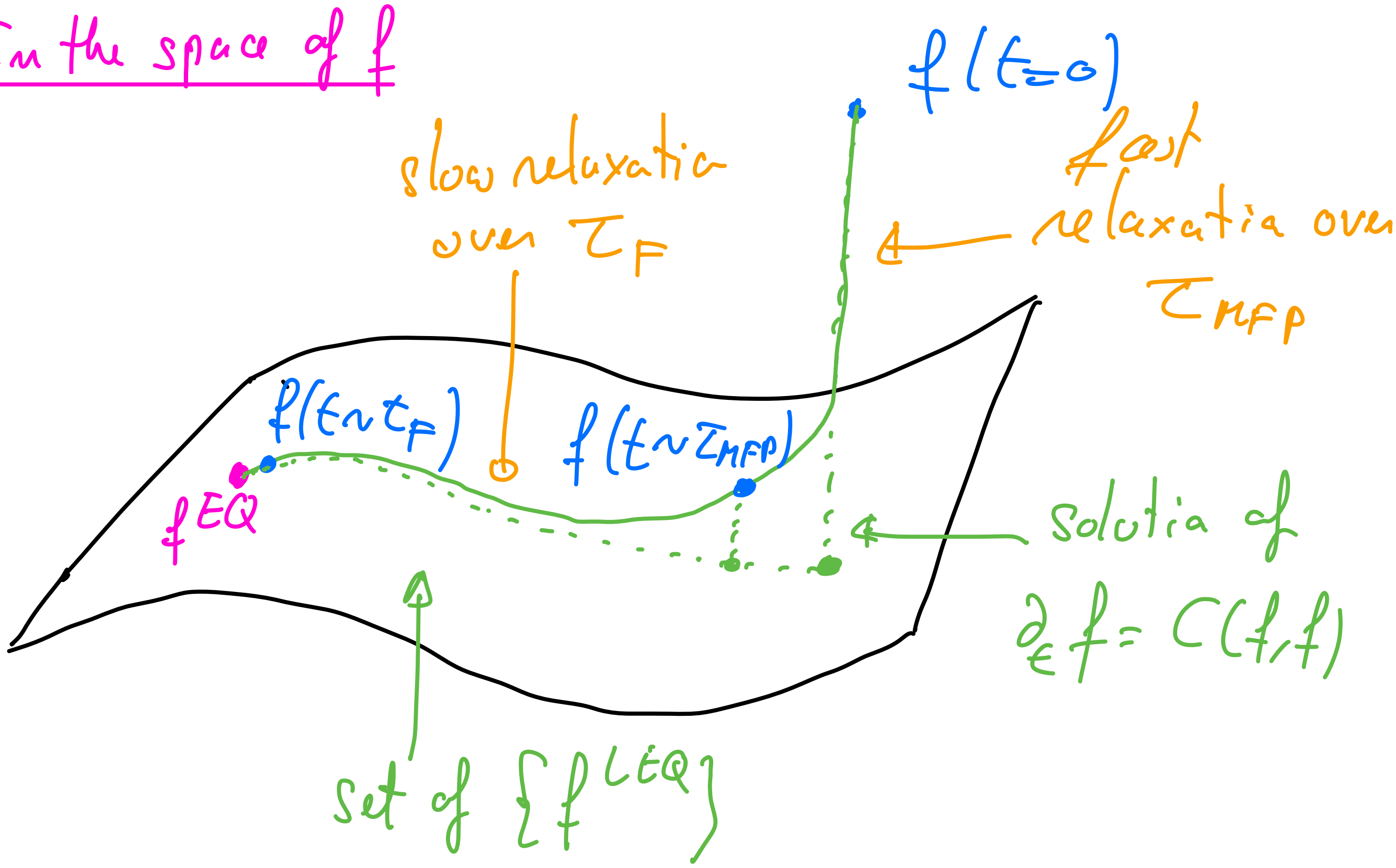
2.4.1) Relaxation of 1 phase space density

$$\partial_t f = -\frac{\vec{p}}{m}\cdot\frac{\partial f}{\partial \vec{q}} + C(f,f)$$

In the space of f



We rewrite the (BE) as $\partial_{\hat{t}} f = -L_F f + \frac{1}{\varepsilon}\hat{C}(f,f)$ (1)

where $L_F f = \tau_F \frac{\vec{p}}{m}\cdot\frac{\partial}{\partial\vec{q}} f \underset{\tau_F \to 0}{\sim} O(1)$

and $\hat{C}(f,f) = \tau_{MFP}\, C(f,f) \underset{\tau_{MFP}\to 0}{\sim} O(1)$

and $t = \hat{t}\,\tau_F$ & $\varepsilon = \frac{\tau_{MFP}}{\tau_F}$

Perturbation theory

$$f(\vec{q},\vec{p},t) = f_0(\vec{q},\vec{p},t) + \varepsilon f_1(\vec{q},\vec{p},t)$$

Order by order

$$\mathcal{O}\left(\tfrac{1}{\varepsilon}\right): \quad 0 = C(f_0, f_0) \Rightarrow f_0 = f^{LEQ}$$

$$\mathcal{O}(1): \quad \partial_{\hat{t}} f_0 + L_1 f_0 = C(f_0, f_1) + C(f_1, f_0)$$

Leading order & slow fields

$$f_0(\vec{q}, \vec{p}, t) = \tilde{\gamma}(\vec{q}, t)\, e^{-\vec{\alpha}(\vec{q},t)\cdot\vec{p} - \beta(\vec{q},t)\left[\frac{\vec{p}^2}{2m} + U(\vec{q})\right]}$$

Can we parametrize f_0 in terms of the conserved fields?

density $\quad n(\vec{q}, t) = \int d\vec{p}\; f(\vec{q}, \vec{p}, t) \qquad$ (d)

momentum $\quad \vec{w}(\vec{q}, t) = \int d\vec{p}\; \vec{p}\, f(\vec{q}, \vec{p}, t) \qquad$ (m)

kinetic energy $\quad K(\vec{q}, t) = \int d\vec{p}\; \frac{\vec{p}^2}{2m} f(\vec{q}, \vec{p}, t) \qquad$ (k)

In principle, $n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2$, $\vec{w} = \vec{w}_0 + \varepsilon \vec{w}_1 + \varepsilon^2 w_2 + \dots$

$$K = K_0 + \varepsilon K_1 + \varepsilon^2 K_2$$

(d, m, k), to order 0, show that n_0, $\vec{w}_0$ & K_0 determine entirely f_0, which is a Gaussian function, hence entirely determined by its normalization & two first moments.

To determine the evolution of f_0, we can thus use the evolution of n, $\vec{w}$ & K $\Rightarrow$ let's build them!

2.4.2) The hydrodynamic equations

Evolution of the density field

$n(\vec{q}) = \int d^3\vec{p}_1 \, f(\vec{q},\vec{p}_1) \Rightarrow \int d^3\vec{p}_1 \, (BE)$ gives

$$\partial_t n + \frac{\partial}{\partial \vec{q}} \cdot \int d^3\vec{p}_1 \, f \frac{\vec{p}_1}{m} = \int d^3\vec{p}_1 d^3\vec{p}_2 \, d^2\sigma \, |\vec{v}_1 - \vec{v}_2| \, (f_1' f_2' - f_1 f_2) \times 1$$

symmetrize the r.h.s by doing $\vec{p}_1 \leftrightarrow \vec{p}_2$ & then $(\vec{p}_1,\vec{p}_2) \leftrightarrow (\vec{p}_1{}',\vec{p}_2{}')$

$$\partial_t n + \frac{\partial}{\partial \vec{q}} \cdot j(\vec{q}) = \frac{1}{4} \int d^3\vec{p}_1 d^3\vec{p}_2 \, d^2\sigma \, |\vec{v}_1 - \vec{v}_2| \, (f_1' f_2' - f_1 f_2) \underbrace{\left[\overbrace{1 + 1}^{\vec{p}_1 \leftrightarrow \vec{p}_2} \underbrace{- 1 - 1}_{\vec{p}_1 \leftrightarrow \vec{p}_1{}'} \right]}_{=0}$$

$$j(\vec{q}) = \int d^3\vec{p} \, \vec{v} \, f(\vec{q},\vec{p},t)$$

$$\Leftrightarrow \quad \boxed{\partial_t n + \frac{\partial}{\partial \vec{q}} \cdot \vec{j}(\vec{q}) = 0} \quad \text{local conservation law}$$

Velocity field: $f(\vec{q},\vec{p},t)$ is locally proportional to the number of particles $\Rightarrow$ so is $\vec{j}(\vec{q})$, one thus defines a local velocity field as $\boxed{j(\vec{q},t) \equiv n(\vec{q},t)\, \vec{u}(\vec{q},t)}$

This can be rewritten as

$$\boxed{\vec{u}(\vec{q},t) = \frac{\int d^3\vec{p} \, \vec{v} \, f(\vec{q},\vec{p},t)}{\int d^3\vec{p} \, f(\vec{q},\vec{p},t)} = \langle \vec{v} \rangle_{f,\vec{q}}}$$

where $\langle \mathcal{O}(\vec{p}) \rangle_{f,\vec{q}} = \int d^3\vec{p}\ \mathcal{O}(\vec{p})\ \rho_{f,\vec{q}}(\vec{p})$

and $\rho_{f,\vec{q}} = \dfrac{f(\vec{q},\vec{p},t)}{\int d^3\vec{p}\ f(\vec{q},\vec{p},t)}$ is the conditional probability density that a particle at $\vec{q}$ has momentum $\vec{p}$.

To lighten notations in this chapter we drop the subscript

$$\langle \dots \rangle_{f,\vec{q}} \to \langle \dots \rangle$$

Evolution equation for the density field

$$\boxed{\partial_t n(\vec{q},t) = -\frac{\partial}{\partial \vec{q}} \cdot \left[\vec{u}(\vec{q},t)\, n(\vec{q},t)\right]} \qquad (1)$$

Material derivatives

$$(1) \iff \partial_t \vec{n} + \vec{u} \cdot \frac{\partial}{\partial \vec{q}} \vec{n} = -n \frac{\partial}{\partial \vec{q}} \cdot \vec{u}$$

$$\iff \boxed{D_t \vec{n}(\vec{q},t) = -n \frac{\partial}{\partial \vec{q}} \cdot \vec{u} = -n\, \partial_{q_\alpha} u_\alpha} \qquad (1')$$

where $\boxed{D_t = \frac{\partial}{\partial t} + \vec{u} \cdot \frac{\partial}{\partial \vec{q}} = \frac{\partial}{\partial t} + u_\alpha \frac{\partial}{\partial q_\alpha}}$ is the material derivative

and summations over repeated indices are implicit.

(1) is nice but not closed $\Rightarrow$ dynamics of $\vec{u}$

Evolution of the velocity field

$$u_\alpha = \frac{1}{n}\int dp\, v_\alpha f$$

$$\partial_t u_\alpha = -\frac{\partial_t n}{n^2}\underbrace{\int dp\, v_\alpha f}_{n u_\alpha} + \frac{1}{n}\int dp\, v_\alpha \partial_t f$$

$$= \frac{\partial_{q_\beta}(u_\beta n)}{n} u_\alpha + \frac{1}{n}\int dp\, v_\alpha\left[-v_\beta \partial_{q_\beta} f\right] + \frac{1}{n}\int dp_1\, dp_2\, d\sigma\, |\vec{v}_1 - \vec{v}_2| \left(f_1' f_2' - f_1 f_2\right) v_{1\alpha}$$

symmetrization

$$= u_\alpha \partial_{q_\beta} u_\beta + \frac{u_\alpha u_\beta}{n}\partial_{q_\beta} n - \frac{1}{n}\partial_{q_\beta}\int dp\, v_\alpha v_\beta f + \frac{1}{4n}\int dp_1\, dp_2\, |\vec{v}_1 - \vec{v}_2|\left(f_1' f_2' - f_1 f_2\right) \underbrace{\left(v_{1,\alpha} + v_{2,\alpha} - v_{1,\alpha}' - v_{2,\alpha}'\right)}_{=0 \text{ since } \vec{P}_{CM} = \vec{P}_{CM}'}$$

$\Rightarrow$ true for any collisional invariant

$$D_t u_\alpha = \partial_t u_\alpha + u_\beta \partial_{q_\beta} u_\alpha = \underbrace{\partial_{q_\beta}(u_\alpha u_\beta) + \frac{u_\alpha u_\beta}{n}\partial_{q_\beta} n}_{\frac{1}{n}\partial_{q_\beta}(n u_\alpha u_\beta)} - \frac{1}{n}\partial_{q_\beta}\left(n \langle v_\alpha v_\beta\rangle\right)$$

$$D_t u_\alpha = -\frac{1}{n}\partial_{q_\beta}\Big[n \underbrace{\left(\langle v_\alpha v_\beta\rangle - \langle v_\alpha\rangle\langle v_\beta\rangle\right)}_{\langle (\underbrace{v_\alpha - \langle v_\alpha\rangle}_{\delta v_\alpha})(\underbrace{v_\beta - \langle v_\beta\rangle}_{\delta v_\beta})\rangle}\Big] = -\frac{1}{n}\partial_{q_\beta}\left[n \langle v_\alpha v_\beta\rangle_c\right]$$

This suggests introducing the pressure tensor

$$P_{\alpha\beta} = n\, m \left\langle (v_\alpha - u_\alpha)(v_\beta - u_\beta)\right\rangle = n\, m \langle \delta v_\alpha\, \delta v_\beta\rangle$$

which yields $\boxed{m D_t u_\alpha = -\frac{1}{n}\partial_{q_\beta} \cdot P_{\alpha\beta} \Leftrightarrow m D_t \vec{u} = -\frac{1}{n}\vec{\nabla}\cdot \bar{\bar{P}}}$ (2)

Comments: * (1) & (2) are like the Navier Stokes equations for a gas.

* To close (1) & (2) to order ε^0, we need to compute $P_{\alpha\beta}$ & thus know f^0. We still miss the second moment of $\vec{p}$ to do that.

Evolution of the kinetic energy:

$$\varepsilon = \left\langle \frac{1}{2} m \, \delta\vec{v}^{\,2} \right\rangle = \frac{1}{2} m \left(\langle \vec{v}^{\,2} \rangle - \vec{u}^{\,2} \right)$$

Painful algebra leads to:

$$\partial_t \varepsilon + u_\alpha \partial_{q_\alpha} \varepsilon = -\frac{1}{n} \partial_\alpha h_\alpha - \frac{1}{n} P_{\alpha\beta} u_{\alpha\beta} \qquad (3)$$

where:

* $u_{\alpha\beta} = \frac{1}{2} \left(\partial_{q_\alpha} u_\beta + \partial_{q_\beta} u_\alpha \right)$ is the strain rate tensor

* $h_\alpha = \frac{n m}{2} \langle \delta v_\alpha \, \delta v_\beta \delta v_\beta \rangle$ is the kinetic energy flux also called heat flux

$\Leftrightarrow \vec{h} = \frac{n m}{2} \langle \delta\vec{v} \, \|\delta\vec{v}\|^2 \rangle$

Proof of Eq (3):

$$\partial_t (n\varepsilon) = \frac{1}{2} m \, \partial_t \int d^3\vec{p} \, f \, (\vec{v}-\vec{u})^2$$

$$= \underbrace{\frac{m}{2} \int d^3\vec{p} \, (\vec{v}-\vec{u})^2 \partial_t f}_{2} + \underbrace{m \int d^3\vec{p} \, f \, (\vec{u}-\vec{v}) \, \partial_t \vec{u}}_{1}$$

$$① = m \partial_t n \Big[\vec{u} \underbrace{\int d^3\vec{p} f}_{=1} - \underbrace{\int d^3\vec{p} f \vec{v}}_{=\vec{u}} \Big] = 0$$

$$② = \frac{m}{2} \int d^3\vec{p} \, (\vec{v}-\vec{u})^2 \Big[-\vec{v}\cdot\frac{\partial}{\partial \vec{q}} f + \underbrace{\int d\vec{p}_2 \, d\sigma \, |v-v_2| \, (f' f_2' - f f_2)}_{} \Big]$$

leads to 0 because $\vec{v}^2$ is a collisional invariants

$$\Rightarrow ② = -\frac{m}{2} \frac{\partial}{\partial q_\alpha} \cdot \Big[\int d^3\vec{p} \, \delta\vec{v}^2 v_\alpha f \Big] + \frac{m}{2} \int d^3\vec{p} \, v_\alpha f \underbrace{\frac{\partial}{\partial q_\alpha} \delta\vec{v}^2}_{2(u_\beta - v_\beta)\partial_{q_\alpha} u_\beta}$$

$$= -\partial_{q_\alpha} \Big[\frac{n m}{2} \langle \delta\vec{v}^2 v_\alpha \rangle \Big] + m (\partial_{q_\alpha} u_\beta) \underbrace{\int d^3\vec{p} \, f \, v_\alpha (u_\beta - v_\beta)}_{}$$

$$= \underbrace{\int d^3\vec{p} \, f \, (v_\alpha - u_\alpha)(u_\beta - v_\beta)}_{-\frac{P_{\alpha\beta}}{m}} + u_\alpha \underbrace{\int d^3\vec{p} \, f \, (u_\beta - v_\beta)}_{=0}$$

$$\Rightarrow \partial_t (n\varepsilon) = -\partial_{q_\alpha} \Big[\frac{n m}{2} \langle \delta\vec{v}^2 v_\alpha \rangle \Big] - (\partial_{q_\alpha} u_\beta) P_{\alpha\beta}$$

$$= n \partial_t \varepsilon + \varepsilon \partial_t n = n \partial_t \varepsilon - \varepsilon \partial_{q_\alpha} (n u_\alpha) = n \partial_t \varepsilon - \partial_{q_\alpha} (n \varepsilon u_\alpha) + n u_\alpha \partial_{q_\alpha} \varepsilon$$

$$\Rightarrow n \partial_t \varepsilon + n u_\alpha \partial_{q_\alpha} \varepsilon = -\partial_{q_\alpha} \Big[\frac{n m}{2} \langle \delta\vec{v}^2 v_\alpha \rangle - \frac{n m}{2} \langle \delta\vec{v}^2 u_\alpha \rangle \Big] - P_{\alpha\beta} u_{\alpha\beta}$$

where we used $(\partial_{q_\alpha} u_\beta) P_{\alpha\beta} = \frac{1}{2} \Big[\partial_{q_\alpha} u_\beta P_{\alpha\beta} + \partial_{q_\beta} u_\alpha \underbrace{P_{\beta\alpha}}_{=P_{\alpha\beta}} \Big] = P_{\alpha\beta} \, \frac{1}{2} (\partial_{q_\alpha} u_\beta + \partial_{q_\beta} u_\alpha)$

$$= P_{\alpha\beta} \, u_{\alpha\beta}$$

All in all, $\partial_t \varepsilon + u_\alpha \partial_{q_\alpha} \varepsilon = -\frac{1}{n} \partial_{q_\alpha} \Big[\frac{n m}{2} \langle \delta\vec{v}^2 \delta v_\alpha \rangle \Big] - \frac{1}{n} P_{\alpha\beta} u_{\alpha\beta} \rightarrow$ Eq (3)

Temperature field

It is useful to define $T(\vec{q},t) = \frac{2}{3}\frac{\varepsilon(\vec{q},t)}{k_B}$ such that $\varepsilon = \frac{3}{2} k_B T$ and

$$\boxed{\partial_t T + u_\alpha \partial_\alpha T = -\frac{2}{3 n k_B}\partial_\alpha h_\alpha - \frac{2}{3 n k_B} P_{\alpha\beta} u_{\alpha\beta}}$$

Closure: We can now close everything to order ε^0.

We can use n, u, T to order ε^0 to characterize f_0 & f_0 to compute P_0 & h_0

which in turn close the dynamics for n, u, T to order ε^0.

Notation: A_0 or A^0 refer to the observable A at order ε^0.

2.4.3) Leading order dynamics

Given $n_0, T_0, \vec{u}_0$, we determine f_0 through

$$\left.\begin{aligned}
&\int f_0 \, d\vec{p} = n_0 \\
&\int \vec{v} f_0 \, d\vec{p} = \vec{u}_0 \\
&\int \frac{m}{2}(\vec{v}-\vec{u}_0)^2 f_0 \, d\vec{p} = n_0(\vec{q})\, \varepsilon_0(\vec{q}) \\
&\qquad = \frac{3 n_0 k_B}{2} T_0(\vec{q})
\end{aligned}\right\}
\quad f_0(\vec{q},\vec{p},t) = \frac{n_0(\vec{q},t)}{\left(2\pi m k_B T_0(\vec{q},t)\right)^{3/2}} e^{-\frac{\left[\vec{p}-m\vec{u}(\vec{q},t)\right]^2}{2 m k_B T(\vec{q},t)}}$$

$$= \frac{n_0}{(2\pi m k_B T_0)^{3/2}} e^{-\frac{m\,\delta\vec{v}^2}{2 k T_0}}$$

Pressure & heat flux

$$P^0_{\alpha\beta} = n_0 m \langle \delta v_\alpha \delta v_\beta \rangle_0 = m \frac{n_0 k_B T_0}{m}\delta_{\alpha\beta} = n_0 k_B T_0 \delta_{\alpha\beta}$$

The pressure is isotropic and obeys the ideal gas law!

$h_\alpha^0 \propto \langle \delta v^2 \delta v_\alpha \rangle = 0$ since it involves an odd of a Gaussian.

Leading order dynamics

$$\partial_t n_0 + \vec{u}_0 \cdot \vec{\nabla} n_0 = - n_0 \vec{\nabla} \cdot \vec{u}_0 \qquad (1)$$

$$m\left[\partial_t \vec{u}_0 + \vec{u}_0 \cdot \vec{\nabla} \vec{u}_0\right] = - \frac{1}{n} \vec{\nabla} \cdot (n_0 k_B T) \qquad (2)$$

$$\partial_t T_0 + \vec{u} \cdot \vec{\nabla} T_0 = - \frac{2}{3} T_0 \vec{\nabla} \cdot \vec{u}_0 \qquad (3)$$

To order ε^0, $f = f^0$ is left invariant by collisions, but transport occurs due to $\{f, H_1\}$, which leads to some evolution of $n_0, \vec{u}_0, T_0$.

Is it enough to equilibrate?